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PLANE PROBLEM OF BENDING OF A SEMI-INFINITE BEAM RESTING ON A LINEARLY DEFORMABLE FOUNDATION

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An exact solution of the problem of bending of a semi-infinite beam lying on an elastic, inhomogeneous half-space $E = E_0 z^\nu$ $(0 \leqslant \nu < 1)$ is obtained. The case when a force or a moment is applied at one end of the beam is solved numerically and an unexpected fact, namely that the absolute magnitude of the maximum relative bending moment increases with increasing ν is discovered.

An exact solution of the three-dimensional problem of bending of a semi-infinite plate lying on a linearly deformable support of a general type, was obtained in [1], where it was already shown that a solution of the corresponding plane problem can be obtained by performing a passage to the limit in the Fourier transform of the solution of the three-dimensional problem. This approach however met with success only in the case when the support was a homogeneous elastic half-space, and even then complicated transformations were needed which could not be applied to the general case. This prompted the author of [2] to provide a solution to the plane problem without reference to the corresponding three-dimensional problem. His method, however, does not yield a solution for the case when the support has the form of a half-space, the modulus of elasticity of which varies according to a power law, not to mention the difficulties encountered in its numerical application.

In the present paper the method of passage to the limit from the three-dimensional problem is used to overcome the difficulties associated with the process of obtaining a solution to the plane problem.

1. Let a semi-infinite plate $(0 \leqslant x < \infty, -\infty < y < \infty)$ of constant cylindrical rigidity D be frictionlessly supported by a linear deformable foundation, for which the settlement of the surface points (kernel of the foundation) caused by a unit force applied at the coordinate origin is given by the formula

$$w_0(r) = \frac{\theta}{2\pi} \int_0^\infty \varphi_0(t) J_0(rt)\, dt, \qquad r = \sqrt{x^2 + y^2} \tag{1.1}$$

where θ is a positive parameter, $J_0(x)$ is the Bessel function of the first kind and the function $\varphi_0(t)$ exhibits the following asymptotic behavior

$$\varphi_0(t) = o(1), \quad t \to 0; \quad \varphi_0(t) = t^\nu [1 + o(1)], \quad t \to \infty \quad (0 \leqslant \nu < 1)$$

We assume that the plate is acted upon by a vertical load $q^+(x, y)$ $(q^+(x, y) \equiv 0$, $x < 0)$, while an additional load $q^-(x, y)$ $(q^-(x, y) \equiv 0,\ x > 0)$ of the form

$$q^\pm(x, y) = q^\pm(x) \cos \lambda y, \qquad \lambda > 0 \quad (-\infty < y < \infty)$$

is applied to the free surface of the support. Then the bending deflections of the plate $w(x, y)$ are equal, within the zone of contact, to the settlement of the surface points of the support and the contact stresses $p(x, y)$ have the form

$$w(x, y);\ p(x, y) = [w_\lambda(x);\ p_\lambda(x)]\cos\lambda y \tag{1.2}$$

The problem of determining the functions $w_\lambda(x)$ and $p_\lambda(x)$ was solved in [1] and the functions were shown to be connected by the following relation:

$$Dw_\lambda(x) = (a_0 + a_1\lambda x)e^{-\lambda x} + y_\lambda(x)$$

$$y_\lambda(x) = \int_0^\infty g(x-\xi)[q^+(\xi) - p_\lambda(\xi)]\,d\xi \tag{1.3}$$

$$g(t) = \frac{1}{4\lambda^3}(1 + \lambda|t|)e^{-\lambda|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm iut}}{(u^2+\lambda^2)^2}\,du$$

Here a_0 and a_1 denote arbitrary real constants which can be obtained from the condition that the plate has a free edge

$$w_\lambda^{(2)}(+0) - \lambda^2\mu w_\lambda(+0) = 0, \qquad w_\lambda^{(3)}(+0) - (2-\mu)\lambda^2 w_\lambda^{(1)}(+0) = 0 \tag{1.4}$$

where μ is the Poisson ratio for the material of the plate.

From (1.3) we can see that the problem reduces to that of determining a single function $p_\lambda(x)$. Let us write the formula for $p_\lambda(x)$ obtained in [1] in the following form:

$$p_\lambda(x) = c^{2\omega}[A_0\Phi_\lambda^{(0)}(x) + A_1\Phi_\lambda^{(1)}(x)] + p_\lambda^*(x), \qquad 2\omega = 3 + \nu$$

$$\Phi_\lambda^{(n)}(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\Psi_\lambda(u)(-iu)^n}{(u+i\lambda)^2}e^{-iux}\,du \qquad (n = 0, 1)$$

$$p_\lambda^*(x) = -\frac{c^{2\omega}}{2\pi}\int_{-\infty}^{\infty}I_\lambda(-u)\Psi_\lambda(u)e^{-iux}\,du \tag{1.5}$$

$$Q^\pm(u) = \int_{-\infty}^{\infty}q^\pm(x)e^{iux}\,dx, \qquad c^{-2\omega} = \theta D$$

$$I_\lambda(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{Q^+(-u)}{(u^2+\lambda^2)^2} - \frac{\varphi_0(\sqrt{u^2+\lambda^2})}{c^{2\omega}\sqrt{u^2+\lambda^2}}\,Q^-(-u)\right]\frac{\Psi_\lambda(u)}{u-z}\,du$$

$$A_0 = a_0 i\lambda\Psi_\lambda(i\lambda) + a_1\lambda[i\Psi_\lambda(i\lambda) + \lambda\Psi_\lambda^{(1)}(i\lambda)]$$

$$A_1 = ia_0\Psi_\lambda(i\lambda) + a_1\lambda\Psi_\lambda^{(1)}(i\lambda)$$

The function $\Psi_\lambda(u)$ appearing in these formulas is regular and different from zero in the upper half-plane ($\operatorname{Im} u > 0$) with the point at infinity excluded, and satisfies the following functional equation:

$$[\varphi_0(\sqrt{u^2+\lambda^2})(u^2+\lambda^2)^{-1/2} + c^{2\omega}(u^2+\lambda^2)^{-2}]^{-1} = \Psi_\lambda(u)\Psi_\lambda(-u) \tag{1.6}$$

We write the solution of this equation which was given in [1], in the more convenient form:

$$\Psi_\lambda(u) = (\lambda - iu)^{1/2(1-\nu)}\chi_1(\lambda, u)\chi_2(\lambda, u)$$

$$-\ln\chi_k(c\lambda, cu) = G_k^+(\lambda, u) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln G_k(\lambda, x)}{x-u}\,dx \qquad (\operatorname{Im} u > 0,\ k = 1, 2)$$

$$G_1(\lambda, u) = 1 + (u^2 + \lambda^2)^{-\omega}$$
$$G_2(\lambda, u) = [1 + c^{-\nu}(u^2 + \lambda^2)^{3/2}\varphi_0(c\sqrt{u^2+\lambda^2})]\,[1 + (u^2+\lambda^2)^{\omega}]^{-1} \tag{1.7}$$

In the second formula of (1.7) that branch of the logarithmic function is chosen, for which the expansion

$$\ln(1+z) = \sum_{k=0} \frac{(-1)^k z^{k+1}}{k+1} \tag{1.8}$$

which shall often be used in the following, holds near the unity. Inserting the expression for $p_\lambda(x)$ given above into (1.3) or into another, equivalent relation [1], we obtain $w_\lambda(x)$. Then we have

$$Dw_\lambda^{(n)}(x) = \frac{d^n}{dx^n}\,[A_0F_\lambda^{(0)}(x) + A_1F_\lambda^{(1)}(x)] + u^{(n)}(x, \lambda) \quad (n = 0, 1, 2, 3, 4).$$

$$F_\lambda^{(m)}(x) = \frac{i}{2\pi}\,\frac{\Psi_\lambda(u)(-iu)^m\,\varphi_0(\sqrt{u^2+\lambda^2})}{(u+i\lambda)^2\sqrt{u^2+\lambda^2}}\,e^{-iux}\,du \quad (m = 0, 1) \tag{1.9}$$

$$u^{(n)}(x,\lambda) = \frac{1}{2\pi}\,\frac{d^n}{dx^n}\int_{-\infty}^{\infty}[Q^-(u) - I_\lambda(-u)\,\Psi_\lambda(u)]\,\frac{\varphi_0(\sqrt{u^2+\lambda^2})}{\sqrt{u^2+\lambda^2}}\,e^{-iux}\,du$$

A solution of the corresponding plane problem of bending of a beam is obtained by performing in the above formulas a passage to the limit with $\lambda \to 0$, in accordance with (1.2). After this the arbitrary constants A_0 and A_1 can be found from the conditions (1.4) at the free edge. These conditions can be written as

$$w^{(n)}(x) = \lim_{\lambda\to 0} w_\lambda^{(n)}(x) = 0 \quad \text{when} \quad x = 0 \quad (n = 2, 3) \tag{1.10}$$

The process of obtaining these constants can be considerably simplified by making use of the following property (*) of the function $y_\lambda(x)$ appearing in (1.3)

$$\left(\frac{d}{dx} - \lambda\right)^2 y_\lambda(x) = \frac{d}{dx}\left(\frac{d}{dx} - \lambda\right)^2 y_\lambda(x) = 0 \quad \text{when} \quad x = 0 \tag{1.11}$$

To prove this property, we shall write the function $g(t)$ defined in (1.3) in the form

$$g(t) = \frac{1}{2\pi}\oint_{C\pm}(u^2+\lambda^2)^{-2}e^{iut}\,du \qquad (t \gtrless 0)$$

The contour C^+ (C^-) represents a closed curve surrounding all poles of the integrand function lying in the upper (lower) half-plane. Using this expression we can show that

$$\left(\frac{d}{dt} - \lambda\right)^2 g(t) = \frac{1}{2\pi}\oint_{C^-}(u - i\lambda)^{-2}e^{iut}\,du = 0 \qquad (t < 0)$$

Similarly we have

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*) This property of $y_\lambda(x)$ representing a particular solution of a Fourier transform of the differential equation describing the bending of a plate was inferred by one of the authors from an unpublished result due to M.G. Krein and presented by him at a seminar at Odessa Civil Engineering Institute in 1955.

$$\frac{d}{dt}\left(\frac{d}{dt}-\lambda\right)^2 g(t)=\frac{1}{2\pi}\oint\limits_{C-}(u-i\lambda)^{-2}iue^{iut}\,du=0 \qquad (t<0)$$

Equation (1.11) now follows from the last two relations. The property (1.11) which has just been proved, makes possible the assertion that the conditions (1.10) at the free edge will hold when $a_0 = a_1 = 0$ or, by virtue of the formulas appearing in (1.5), when $A_0 = A_1 = 0$.

Indeed, for this to be true it is sufficient that

$$u^n(x,0)=0 \quad \text{when } x=0 \quad (n=2,3)$$

The validity of the latter can be shown by comparing the right hand sides of the formulas for $w_\lambda(x)$ appearing in (1.3) and (1.9), with $\lambda \to 0$ and $a_0 = a_1 = 0$ and with the property (1.11) taken into account.

Taking the above argument into consideration, we obtain the following expressions for the bending deflections $w(x)$ and the contact stresses $p(x)$:

$$Dw(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}[Q^-(u)-I_0(-u)\Psi_0(u)]\,u^{-1}\varphi_0(u)e^{-iux}\,du$$

$$p(x)=-\frac{c^{2\omega}}{\pi}\int\limits_{-\infty}^{\infty}I_0(-u)\,\Psi_0(u)e^{-iux}\,du$$

$$\Psi_0(u);\ I_0(u)=\lim_{\lambda\to 0}[\Psi_\lambda(u);\ I_\lambda(u)]$$

As we see, the problem reduces to that of determining the functions $\Psi_0(u)$ and $I_0(u)$. . To find the first function, e.g., we clearly must perform the limiting passage as $\lambda \to 0$ in the right-hand side of the second formula of (1.7). This presents no difficulties when $k = 2$, but becomes almost impossible when $k = 1$. It was done in [1], but only for the case $\nu = 0$ and the process involved complex transformations of the integral in (1.7).

Below we give a method of transforming the integral of the type (1.7) into the form suitable for executing the limiting passage mentioned above and for application of numerical methods to the results obtained.

The limiting passage $\lambda \to 0$ in (1.6) needed for the determination of $I_0(u)$ is connected intrinsically with the need for additional information concerning the support and the load, and will therefore be realized for a support of a specific type.

2. It can be verified that in the present case we have

$$G_k(\lambda, \pm\infty)=1, \quad G_k(\lambda,-x)=G_k(\lambda,x)$$

$$\ln G_k(\lambda,x)=O(|x|^{-\alpha}) \qquad |x|\to\infty \ (k=1,2;\ \alpha\geqslant 1) \tag{2.1}$$

Moreover, the logarithmic derivatives of the functions $G_k(\lambda, x)$ are integrable everywhere in the interval $(-\infty < x < \infty)$. Integration by parts of (1.7), use of the properties (2.1) of $G_k(\lambda, x)$ and subsequent reduction to a semi-infinite integral, yields

$$G_k^+(\lambda,z)=\frac{1}{2\pi i}\int\limits_0^\infty \ln\frac{z+x}{z-x}\,\frac{dG_k}{G_k(\lambda,x)} \qquad (k=1,2) \tag{2.2}$$

Using now the bilinear transformation $z = -i(\zeta+1)(\zeta-1)^{-1}$ we can map the upper half of the z-plane into the unit circle in the ζ-plane.

Then the formula (2.2) becomes

$$g_k(\lambda,\zeta) = G_k^+\left(\lambda, -i\frac{\zeta+1}{\zeta-1}\right) = \tag{2.3}$$

$$= \frac{1}{2\pi i}\int_0^\infty \ln\left(1+\zeta\frac{i-x}{i+x}\right)\frac{dG_k}{G_k(\lambda,x)} - \frac{1}{2\pi i}\int_0^\infty \ln\left(1+\zeta\frac{i+x}{i-x}\right)\frac{dG_k}{G_k(\lambda,x)} + g_{k,0}(\lambda)$$

$$g_{k,0}(\lambda) = -\frac{1}{\pi}\int_0^\infty \operatorname{arctg} x\,\frac{dG_k}{G_k(\lambda,x)} \tag{2.4}$$

Combining the two integrals in (2.3) into a single integral defined on the whole of the real axis and utilizing (1.8), we obtain

$$g_k(\lambda,\zeta) = \sum_{n=0}^{\infty} g_{k,n}(\lambda)\zeta^n,$$

$$g_{k,n}(\lambda) = -\frac{1}{2\pi i n}\int_{-\infty}^{\infty}\left(\frac{x-i}{x+i}\right)^n \frac{dG_k}{G_k(\lambda,x)} \quad \begin{pmatrix} k=1,2 \\ n=1,2,\dots\end{pmatrix} \tag{2.5}$$

When $k = 1$, the coefficients $g_{1,n}(\lambda)$ of the last expansion, by virtue of (1.7), have the form

$$g_{1,0}(\lambda) = \frac{2\omega}{\pi}\int_0^\infty r_\lambda(x)\,x\operatorname{arctg} x\,dx$$

$$g_{1,n}(\lambda) = \frac{\omega}{\pi i n}\int_{-\infty}^{\infty}\left(\frac{x-i}{x+i}\right)^n x r_\lambda(x)\,dx$$

$$r_\lambda^{-1}(x) = [1+(x^2+\lambda^2)^\omega](x^2+\lambda^2)$$

Following [3], we now transform the above formula using the contour integration. Omitting the intermediate steps, we give the final result

$$g_{1,n}\lambda/c = \frac{1}{n}\left[\omega\left(\frac{\lambda-c}{\lambda+c}\right)^n + c^{2\omega}\sum_{i=0}^{1}\left(\frac{\alpha_j-ic}{\alpha_j+ic}\right)b_j^{-2\omega}\right] + h_n(\lambda) \quad (n=1,2,\dots)$$

$$h_n(\lambda) = \frac{2\omega c^2}{n\pi}\int_{\lambda/c}^{\infty}\left(\frac{x-1}{x+1}\right)^n \frac{R_\lambda(x)\,x}{c^2x^2-\lambda^2}\,dx$$

$$R_\lambda(x) = \frac{(x^2-c^{-2}\lambda^2)^\omega \cos{}^1/{}_2\pi\nu}{1+2(x^2-c^{-2}\lambda^2)^\omega \sin{}^1/{}_2\pi\nu + (x^2-c^{-2}\lambda^2)^{2\omega}} \tag{2.6}$$

$$\alpha_j = \sqrt{b_j^2-\lambda^2}, \qquad \operatorname{Im}\alpha_j > 0 \quad (j=1,2); \qquad b_{1,2} = c\exp(\pm i\gamma)\left(\gamma = \frac{\pi}{3+\nu}\right)$$

When $k=2$, we perform the substitution $x=\operatorname{tg}{}^1/{}_2\varphi$ in the integrals of (2.4) and (2.5) and obtain

$$g_{2,0}(\lambda) = -\frac{1}{2\pi}\int_0^\pi \varphi\,dg_\lambda^*(\varphi) = \frac{1}{2\pi}\int_0^\pi g_\lambda^*(\varphi)\,d\varphi$$

$$g_{2,n}(\lambda) = \frac{(-1)^{n+d}}{n\pi}\int_0^\pi \sin n\varphi\,dg_\lambda^*(\varphi) = \frac{(-1)^n}{\pi}\int_0^\pi g_\lambda^*(\varphi)\cos n\varphi\,d\varphi \quad (n=1,2,\dots)$$

$$g_\lambda^*(\varphi) = \ln G_2(\lambda, \operatorname{tg}{}^1/{}_2\varphi)$$

Using (2.5) we now make $\lambda \to 0$ in the expression for $\Psi_\lambda(u)$ given in (1.7). Summing the resulting weakly convergent series and using (1.8), we obtain

$$\Psi_0(icu) = \frac{2^\omega e^{1/2(1-\nu)}}{(1+u)^\omega} u^2 \left[1 + \left(\frac{1-u}{1+u}\right)^2 \mathrm{tg}^2 \frac{\gamma(1+\nu)}{4}\right]^{-1} \exp\left[-H(u) - H_1(u)\right]$$

$$H(u);\ H_1(u) = \sum_{n=0}^{\infty} [h_n;\ g_n] \left(\frac{u-1}{u+1}\right)^n \qquad (2.7)$$

The coefficients of the last expansion are given by

$$g_0 = -\frac{1}{2\pi}\int_0^\pi \varphi\, dg_0^*(\varphi) = \frac{1}{2\pi}\int_0^\pi g_0^*(\varphi)\, d\varphi$$

$$g_n = \frac{(-1)^{n+1}}{n\pi}\int_0^\pi \sin n\varphi\, dg_0^*(\varphi) = \frac{(-1)^n}{\pi}\int_0^\pi g_0^*(\varphi)\cos n\varphi\, d\varphi \qquad (2.8)$$

$$g_0^*(\varphi) = \ln\left[1 + c^{-\nu}\,\mathrm{tg}^3\, {}^1\!/_2\varphi\,\varphi_0(\mathrm{tg}\, {}^1\!/_2\varphi c)\right] - \ln\left(1 + \mathrm{tg}^\omega\, {}^1\!/_2\varphi\right)$$

$$h_0 = \frac{2\omega}{\pi}\int_0^\infty \frac{\mathrm{arctg}\, x\, dx}{x(1+x^{2\omega})}, \qquad h_n = \frac{2\omega}{n\pi}\int_0^\infty \left(\frac{x-1}{x+1}\right)^n \frac{R_0(x)}{x}\, dx \qquad (n = 1, 2, \ldots)$$

The last two of the above formulas can be written in the form

$$h_0 = \frac{2\omega}{\pi}\int_0^1 \left[\frac{\mathrm{arctg}\, x}{x} + x^{2+\nu}\,\mathrm{arctg}\, x\right]\frac{dx}{1+x^{2\omega}}$$

$$h_{2m} = \frac{2\omega}{m\pi}\int_0^1 \left(\frac{1-x}{1+x}\right)^{2m} \frac{R_0(x)}{x}\, dx, \qquad h_{2m-1} \equiv 0 \qquad (m = 1, 2, \ldots)$$

more suitable for numerical work (*) and obtained by dividing the interval of integration into two subintervals, (0,1) and $(1, \infty)$. The identity $h_{2m-1} \equiv 0$ $(m = 1, 2, \ldots)$ implies the following property

$$H(u) = H(1/u) \qquad (2.9)$$

which is useful for computations, and will be employed later.

We can see from (2.8) that the coefficients g_n are the Fourier coefficients of the function $g_0^*(\varphi)$ or of its derivative. For this reason the coefficients can be conveniently computed using the method of trigonometric interpolation [4].

3. Let us apply the above formulas to the case of a support consisting of an elastic half-space $z \geqslant 0$, the modulus of elasticity $E = E_0 z^\nu$ $(0 \leqslant \nu < 1)$ of which varies with depth and the kernel of which can be obtained from (1.1) by putting [5]

$$\theta = \frac{(1-\mu_0^2)\,\Gamma({}^1\!/_2 - {}^1\!/_2\nu)\, qC}{2\sqrt{\pi}\, E_0 \Gamma(1 + {}^1\!/_2\nu)(1+\nu)} \sin\frac{\pi q}{2}, \qquad \varphi_0(t) = t^\nu \qquad (3.1)$$

$$\frac{q^2}{1+\nu} = 1 - \frac{\nu\mu_0}{1-\mu_0}, \qquad \frac{C}{\Gamma[1 + {}^1\!/_2(1+\nu+q)]} = \frac{2\Gamma[1 + {}^1\!/_2(1+\nu-q)]}{\Gamma(2+\nu)}$$

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*) Computations performed below have shown the rapid convergence of the series defining $H(u)$: the choice of $m = 5$ was sufficient to obtain a value accurate to three significant figures.

where μ_0 is the Poisson ratio for the support material. Moreover, we shall assume that the additional load is absent $(q^-(x) \equiv 0)$ and that the load acting on the plate is given by

$$q(x, y) = \delta(x - b)\cos\lambda y, \quad b > 0$$

where $\delta(x)$ is the Dirac delta function. Then the integral $I_\lambda(z)$ appearing in (1.5) will assume the form

$$I_\lambda(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-ibu}\Psi_\lambda(u)}{(u^2+\lambda^2)^2(u-z)}\,du \tag{3.2}$$

Since we are considering a particular type of support, the above formula as well as (1.5) and (1.9) can be further transformed using the methods of contour integration. This requires the knowledge of the singular points of the function $\Psi_\lambda(u)$ in the lower half-plane. To find them we proceed to write the function in the following form using (1.6) and (3.1):

$$\Psi_\lambda(u) = (u^2+\lambda^2)^2 f_\lambda(u)\Psi_\lambda^{-1}(-u), \qquad f_\lambda(u) = [c^{2\omega} + (x^2+\lambda^2)^\omega]^{-1}$$

This shows that in the lower half-plane $\Psi_\lambda(u)$ has one branch point $u = -i\lambda$ and the poles $u = -\alpha_j$ $(j = 1, 2)$ [3] determined by the formula appearing in (2.6). The residues and the difference in the values of the function at the edges of the cut $(-i\lambda, -i\infty)$ are given by

$$\operatorname{Res}[\Psi_\lambda(u)]_{u=-\alpha_j} = -b_j^{3-\nu}[2\omega\alpha_j\Psi_\lambda(\alpha_j)]^{-1} \qquad (j = 1, 2)$$

$$\Psi_\lambda(-is+0) - \Psi_\lambda(-is-0) = -2i(s^2-\lambda^2)^2 c^{-2\omega}R_\lambda(s/c)\Psi^{\lambda-1}(is) \tag{3.3}$$

Using the latter we now transform (3.2) be deforming the path of integration into a loop [6, 7] enveloping the ray $(-i\lambda, -i\infty)$. This yields

$$I_\lambda(z) = I_\lambda^*(z) - \Psi_\lambda(z)e^{-ibz}(z^2+\lambda^2)^{-2} \tag{3.4}$$

$$\frac{I_\lambda^*(z)}{c^{-2-\nu}} = -\frac{i}{\pi}\int_{\lambda/c}^{\infty}\frac{R_\lambda(t)e^{-bct}}{\Psi_\lambda(ict)(z+ict)}\,dt - \frac{c^{2+\nu}}{2\omega}\sum_{j=1}^{2}\frac{b_j^{-(1+\nu)}e^{i\alpha_j b}}{\alpha_j\Psi_\lambda(\alpha_j)(\alpha_j+z)}$$

Inserting now (3.4) into (1.5) and (1.9), changing the order of integration and transforming the inner integral in the manner similar to that already employed, we arrive at

$$u^{(n)}(x,\lambda) = -\frac{c}{\pi}\int_{\lambda/c}^{\infty}\frac{I_\lambda^*(ict)}{\Psi_\lambda(ict)}R_\lambda(t)(-ct)^n e^{-ctx}\,dt - \tag{3.5}$$

$$-\,i(2\omega)^{-1}\sum_{j=1}^{2} b_i^2\frac{(i\alpha_j)^n e^{i\alpha_j x}I_\lambda^*(\alpha_j)}{\alpha_j\Psi_\lambda(\alpha_j)} + Du_\infty^{(n)}(x-b,\lambda) \qquad (n = 0, 1, 2, 3)$$

$$u_\infty(x,\lambda) = \frac{c}{\pi}\int_{\lambda/c}^{\infty}R_\lambda(t)(c^2t^2-\lambda^2)^{-2}e^{-ct|x|}\,dt + \frac{i}{2\omega}\sum_{j=1}^{2}\frac{e^{i\alpha_j|x|}}{\alpha_j b_j^2}$$

$$p_\lambda^*(x) = \frac{c}{\pi}\int_{\lambda/c}^{\infty}(c^2t^2-\lambda^2)R_\lambda(t)\Psi_\lambda^{-1}(ict)I_\lambda^*(ict)e^{-ctx}\,dt -$$

$$-\frac{ic^{2\omega}}{2\omega}\sum_{j=1}^{2}\frac{b_j^{3-\nu}e^{i\alpha_j x}}{\alpha_j\Psi_\lambda(\alpha_j)}I_\lambda^*(\alpha_j)+p_\infty(x-b,\lambda)$$

$$p_\infty(x,\lambda)=-\frac{c}{\pi}\int_{\lambda/c}^{\infty}R_\lambda(t)e^{-ct|x|}dt+\frac{i}{2\omega}\sum_{j=1}^{2}\frac{e^{i\alpha_j|x|}}{\alpha_j b_j^{1+\nu}}$$

Setting $\lambda=0$ in (3.4) and (3.5) and taking into account (2.7), we obtain

$$cM(cx)=M^*(\xi)=\frac{1}{\pi}\int_0^\infty S(t)\left[e^{-\beta t}f_1(t,\xi)-I(t;\beta)t^2e^{-\xi t}\right]dt$$

$$-f_2(\xi,\beta)+M_\infty(\xi-\beta)$$

$$I(t;\beta)=(2^\omega\pi)^{-1}\int_0^\infty S(z)e^{-\beta z}(t+z)^{-1}dz-f_3(t;\beta)$$

$$S(t)=R_0(t)t^{-2}(1+t)^\omega\left[1+\left(\frac{1-t}{1+t}\right)^2\operatorname{tg}^2\left({}^1/_4\gamma(1+\nu)\right)\right]\exp H(t)$$

$$f_1(t;\xi)=2^{1-\omega}a\frac{\cos(\delta-\gamma+\xi\cos\gamma)+t\sin(\delta+\xi\cos\gamma)}{(1+2t\sin\gamma+t^2)\exp(\xi\sin\gamma)} \qquad (3.6)$$

$$f_2(\xi,\beta)=\{\sin\left[{}^1/_4\gamma(5-\nu^2)+(\beta+\xi)\cos\gamma\right]+$$
$$+\sin^{-1}\gamma\cos\left[(1+\nu)\gamma+(\beta-\xi)\cos\gamma\right]\}a^2\exp\left[-(\xi+\beta)\sin\gamma\right]$$

$$f_3(t;\gamma)=2a\frac{\cos(\gamma+\varepsilon-\beta\cos\gamma)-t\sin(\varepsilon-\beta\cos\gamma)}{(1+2t\sin\gamma+t^2)\exp(\beta\sin\gamma)},\qquad \varepsilon={}^1/_8\gamma(11+4\nu+\nu^2)$$

$$a=(2\omega)^{-1}\cos^{1/2(\nu-5)}{}^1/_4\gamma(1+\nu)\cos{}^1/_2\gamma(1+\nu)\exp H(\alpha_1/c),$$

$$\delta={}^1/_8\gamma(1-\nu)(5-\nu)$$

Here we have introduced the dimensionless abscissa $\xi=cx$ and the dimensionless distance $\beta=cb$ between the point of application of the force and the beam end. Following [1] we shall call the quantity $M^*(\xi)$ the reduced bending moment. Similarly we shall call $p^*(\xi)$ and $Q^*(\xi)$ the reduced contact stress and the reduced transverse force, connected with the true $p(x)$, $Q(x)$ and $M^*(\xi)$ by the formulas

$$Q^*(\xi)=Q\left(\frac{\xi}{c}\right),\ p^*(\xi)=\frac{1}{c}p\left(\frac{\xi}{c}\right),\quad Q^*(\xi)=\frac{dM^*}{d\xi},\quad p^*(\xi)=\frac{d^2M^*}{d\xi^2}$$

The symbol $M_\infty(\xi)$ appearing in (3.6) denotes the reduced bending moment in an anfinite beam acted upon by a unit force at $x=0$ and determined bv the formula

$$M_\infty(\xi)=\frac{\sin\pi\omega}{\pi}\int_0^1\frac{t^{1+\nu}e^{-|\xi|t}+t^{3+\nu}e^{-|\xi|/t}}{1+2t^{3+\nu}\cos\pi\omega+t^{6+2\nu}}dt-\omega^{-1}\exp(-|\xi|\sin\gamma)\sin(|\xi|\cos\gamma-\gamma) \qquad (3.7)$$

which coincides with that given in [3].

As we know, in the analysis of the semi-infinite beams the most interesting case encountered is that, in which the end of the beam is acted upon by a force or a moment. We arrive at the case of force loading by setting in (3.6) $\beta=0$, and at the case of moment loading by differentiating (3.6) with respect to β and setting $\beta=0$ again.

A different, simpler method can be used to obtain formulas for computing the forces in end-loaded beams. First we set in (1.9) $u(x,\lambda)=0$, then perform a passage to the limit with $\lambda\to 0$. In other words, the reduced bending moment in a semi-infinite end-loaded beam can be obtained using the formula

$$M^*(\xi) = -c \lim_{\lambda\to 0} [A_0 F_\lambda^{(2)}(\xi/c) + A_1 F_\lambda^{(3)}(\xi/c)] \quad (3.8)$$

To obtain formulas suitable for computation we transform the expression for $F_\lambda^{(n)}(x)$ appearing in (1.9) by applying the methods of contour integration prior to the limiting passage $\lambda \to 0$. As the result we have

$$F_\lambda^{(n)}(x) = -\frac{ic}{\pi}\int_{\lambda/c}^{\infty} \frac{(-ct)^n e^{-ctx} R_\lambda(t)}{(ct-\lambda)^2 \Psi_\lambda(ict)} dt -$$

$$-(2\omega)^{-1}\sum_{j=1}^{2} \frac{b_j^2 (i\alpha_j)^n e^{i\alpha_j x}}{\alpha_j \Psi_\lambda(\alpha_j)(i\lambda-\alpha_j)^2} \quad (n = 0, 1, 2, 3, 4)$$

Setting $\lambda = 0$, in the above formulas we obtain, of the basis of (3.8) and using (2.9),

$$M^*(\xi) = -\sum_{j=0}^{1} B_j[(-1)^j J^{(j)}(\xi) + \varphi^{(j)}(\xi)] \quad (3.9)$$

$$\varphi^{(j)}(\xi) = 2a\frac{d^j}{d\xi^j}\{\cos[{}^1/_8\gamma(1-\nu^2) + \xi\cos\gamma]\exp(-\xi\sin\gamma)\}$$

$$B_0 = ic^\omega A_0, \qquad B_1 = c^{1+\omega}A_1 i$$

$$J^{(j)}(\xi) = \frac{-1}{2^\omega \pi}\int_0^1 [t^j e^{-\xi t} + t^{2-\omega-j} e^{-\xi/t}]\, S(t)\, dt$$

In addition, we have the following property:

$$dJ^{(j)}/d\xi = -J^{(j+1)}(\xi) \qquad (j = 0, 1, 2, 3)$$

It can easily be shown that the arbitrary constants in (3.9) are given by

$$\sum_{n=0}^{1} B_n[(-1)^{(n)} J^{(n)}(0) + \varphi^{(n)}(0)] = 1 - m$$

$$(m = 1, 2)$$

$$\sum_{n=0}^{1} B_n[(-1)^{n+1} J^{(n+1)}(0) + \varphi^{(n+1)}(0)] = 2 - m$$

The case $m = 1$ corresponds to a beam loaded by a unit force, while $m = 2$ corresponds to the unit moment loading.

The formula (3.9) was used to calculate the values of the reduced bending moment as well as the transverse force $Q^* = dM^*/d\xi$ and the contact stress $p^* = d^2M^*/d\xi^2$ for both,the case of a beam loaded with a unit force (Table 1), and with a unit clockwise moment (Table 2). It must be remembered here that the figures given represent the decimal parts of the values. The integral parts are equal to zero except the ones indicated by asterisks where the integral parts are equal to unity. The negative values are indicated by a bar on top.

The data given in tables were computed for three values of ν ($\nu = 0.1, 0.5$ and 0.9) and show that in the case of $\nu = 0.1$ the results agree, as expected, with those of [1]. The major result emerging from these computations however.is the discovery of an unexpected fact, namely that the absolute magnitude of the maximum bending moment increases with increasing parameter ν describing the rigidity of the support (Table 1). This induced the authors to explain a similar fact encountered in the case of an infinite beam, for which a simpler formula (3.7) defining the reduced bending moment under the action of a concentrated unit force applies. The values of this moment taken from the diploma

TABLE 1

	$-M^*$			Q^*			P^*		
ξ \ ν	0.1	0.5	0.9	0.1	0.5	0.9	0.1	9.5	0.9
0.0	0	0	0	$\bar{0}$*	$\bar{0}$*	$\bar{0}$*	∞	∞	∞
0.2	131	154	170	$\overline{478}$	$\overline{608}$	$\overline{714}$	309*	348*	253*
0 4	205	252	289	$\overline{273}$	$\overline{382}$	$\overline{488}$	805	953	019*
0.6	245	311	368	$\overline{141}$	$\overline{217}$	$\overline{304}$	546	705	822
0.8	264	342	414	$\overline{049}$	$\overline{096}$	$\overline{158}$	377	521	650
1.0	267	351	433	013	$\overline{006}$	$\overline{043}$	257	378	501
1.2	260	346	433	056	058	044	169	265	373
1.4	246	330	417	082	101	108	104	176	264
1.6	228	268	391	098	145	151	056	106	174
1.8	207	259	358	106	159	178	021	052	101
2.0	186	249	321	107	151	193	$\overline{004}$	012	043
3.0	091	112	137	075	111	153	$\overline{044}$	$\overline{065}$	$\overline{085}$
4.0	037	032	027	036	051	069	$\overline{031}$	$\overline{050}$	$\overline{073}$

TABLE 2

	M^*			$-Q^*$			P^*		
ξ \ ν	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
0.0	0*	0*	0*	0	0	0	$-\infty$	$-\infty$	$-\infty$
0.2	945	966	979	397	280	191	$\overline{774}$	$\overline{835}$	$\overline{766}$
0.4	855	897	928	493	403	317	$\overline{265}$	$\overline{431}$	$\overline{506}$
0.6	753	809	856	520	463	398	$\overline{022}$	$\overline{189}$	$\overline{306}$
0.8	649	714	771	509	484	443	112	$\overline{029}$	$\overline{150}$
1.0	550	618	680	479	478	460	188	078	$\overline{029}$
1.2	459	524	588	436	455	456	227	148	060
1.4	376	436	499	389	420	437	242	191	125
1.6	303	326	414	341	380	408	242	214	168
1.8	240	285	336	293	336	371	231	222	195
2.0	186	222	266	249	292	331	214	219	208
3.0	027	029	037	087	109	137	111	136	160
4.0	$\overline{018}$	$\overline{028}$	$\overline{036}$	015	018	024	039	053	070

TABLE 3

	M_∞								
ν \ ξ	0.0	0.2	0.4	0 8	1.0	1.4	2.0	3.0	4.0
0.1	380	288	210	093	051	$\overline{006}$	$\overline{046}$	$\overline{051}$	$\overline{034}$
0.5	365	273	194	076	034	$\overline{023}$	$\overline{060}$	$\overline{056}$	$\overline{032}$
0.9	356	263	184	064	022	$\overline{035}$	$\overline{070}$	$\overline{058}$	$\overline{027}$

thesis of V. V. Vorotyntsev are given in Table 3, which shows that the maximum positive (stretching of the lowest filament) reduced moment decreases with increasing ν, while the numerical value of the maximum negative moment increases just as in the case of a semi-infinite beam discussed above.

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SHOCK WAVE PROPAGATION IN ELASTIC-PLASTIC MEDIA

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It is shown that neutral shock waves, on which the plastic deformations are continuous, and waves on which the plastic deformations are discontinuous, can exist in ideal and hardened elastic-plastic media. Conditions for the existence of waves of the second kind are written down, the velocities of all the mentioned waves are determined in ideally plastic bodies for arbitrary convexity of the flow and Tresca conditions, and in hardened bodies for kinematic and isotropic hardening. Relationships are obtained for the discontinuities upon passage through the wave surface.

The behavior of shock waves during propagation under Mises and Tresca flow conditions is investigated by using the kinematic second-order compatibility conditions. It is shown that the shock wave intensity varies according to laws of geometric optics.

Questions of shock wave propagation in elastic-plastic media have been examined in [1 - 3]. Relationships on the shock waves in hardened elastic-plastic bodies have been derived under the assumption that simple loading occurs on the shock [1]. Reltionships on shock waves in plane ideally elastic-plastic bodies have been obtained in [2] by using the theory of generalized functions. The